Abstract—Hamilton-Jacobi (HJ) reachability provides a flexible framework for the verification of safety in robotic systems: it accounts for nonlinear system dynamics and provides safety-preserving controllers. However, computational scalability limits its direct application to systems of less than five continuous state dimensions. To alleviate this computational burden, system decomposition methods have been proposed; however, safety guarantees are lost in situations involving “leaking corners”, which arise when there are conflicting controls between subsystems. In this paper, a coupled HJ formulation is presented, which addresses leaking corners and guarantees safety, while incorporating dimensionality reduction. We demonstrate our method in two examples, one of which is a vehicle obstacle avoidance problem with a 5D car model, whose HJ computation was previously considered to be intractable.

I. INTRODUCTION

Reachability analysis [1]–[6] has been applied extensively to verify safety properties and generate safe controllers in robotics and control systems. The Hamilton-Jacobi (HJ) formulation of reachability is popular because it applies to general nonlinear systems with control and disturbances, and computes numerically convergent reachable sets and globally optimal control policies [7], [8]. It has been utilized in last-resort collision avoidance [7], [9]–[11], robust trajectory planning [12]–[14], unmanned air traffic management [15]–[18], and other practical problems [19]–[21]. However, the biggest challenge of HJ formulation is the “curse of dimensionality”: the computational complexity is exponential with respect to the dimension of the continuous state.

In this paper, we focus on addressing the computational complexity challenge and provide conservative approximations of the backward reachable set (BRS) and tube (BRT). The BRS is the set of initial states from which all admissible trajectories inevitably arrive at dangerous configurations at the final time of a time horizon, and the BRT is the set of initial states from which all admissible trajectories inevitably arrive at dangerous configurations within a time horizon.

To achieve dimensionality reduction in reachability analysis, recent methods have used decomposition techniques for specific classes of problems. For example, the authors in [22] and [23] propose conservative approximations of BRSs obtained from manipulating projections of sets. If the system has a terminal integrator as in [24], the integrator state can sometimes be removed from the HJ computation without introducing conservatism. One of the themes in these methods, as well as in other methods such as [25], [26], is the computation of multiple BRSs in lower-dimensional subspaces and the subsequent combination of them into one full-dimensional BRS. These approaches work well for guaranteeing safety while significantly reducing computation time if the optimal control for the full-dimensional BRS is one of the optimal controls for the multiple BRSs in lower-dimensional subspaces. If not, for states located near the intersection of the zero level sets of the lower-dimensional BRSs, none of the optimal controls from the lower-dimensional BRSs are optimal for the full-dimensional BRS. These states are called “leaking corners”, a fundamental problem in game theory and optimal control [24], [27], [28].

Contributions: We propose a novel approach for computing BRSs that achieves dimensionality reduction while maintaining safety guarantees for a new class of problems. These problems previously suffered from the leaking corner issue, which prevented methods such as [24]–[26] from making safety guarantees. The key insight of our approach is to solve a set of coupled HJ PDEs, rather than independent HJ PDEs. We achieve dimensionality reduction via an approach similar to [24]. By keeping track of multiple HJ PDEs at the same time, we synthesize a common controller to prevent control conflicts. We validate our method numerically in two examples: a double integrator and a five-dimensional (5D) car collision avoidance problem. In the 5D example, computation time was reduced from 32.5 hours to 41 seconds by the combination of our proposed method and the one in [25].

Organization: In Section II, we present some background in HJ reachability and system decomposition, and state the problem to be solved. In Section III, we first consider the leaking corner problem in the full dimensional problem, and then demonstrate dimensionality reduction for a special class of system dynamics. In Section IV, we present the numerical validation of our theory through two examples. In Section V, we conclude and provide some future directions.

II. PRELIMINARIES

In this paper, we consider a system with state $z \in \mathbb{R}^n$ whose dynamics are described by the following ODE:

$$\dot{z} = \frac{dz}{ds} = f(z,u), \quad s \leq 0.$$ (1)

Here, we use $t < 0$ as the initial time and 0 as the final time, so $s$ is between $t$ and 0.

The control is denoted by $u(s) \in U$, with the control function $u(\cdot) \in \mathbb{R}^n$ being drawn from the set of measurable functions. The dynamics $f: \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ is assumed to be uniformly continuous, bounded, and Lipschitz continuous in $z$ for fixed $u$; given $u(\cdot) \in U$, there exists a unique trajectory solving (1) [29]. We denote trajectories of (1) starting from state $z$ at time $t$ under control $u(\cdot)$ as $\zeta(s;z, t, u(\cdot)) : [t, 0] \rightarrow \mathbb{R}^n$. $\zeta$ satisfies (1) with initial state $z$.

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A. Safety via Minimal BRS and BRT

The target set, denoted $\mathcal{T}$, represents dangerous configurations such as collisions with obstacles. The minimal BRS is the set of initial states from which all controls $u(\cdot)$ take the system into the $\mathcal{T}$ at the final time $s = 0$. Mathematically, the minimal BRS is defined as follows.

$$A(\mathcal{T}, t) := \{ z : \forall u(\cdot) \in U, \zeta(0; z, t, u(\cdot)) \in \mathcal{T} \}$$  \hspace{1cm} (3)

The minimal BRT has a similar definition, except the entire interval $s \in [t, 0]$ is considered:

$$A(\mathcal{T}, t) := \{ z : \forall u(\cdot) \in U, \exists t \in [t, 0], \zeta(s; z, t, u(\cdot)) \in \mathcal{T} \}. \hspace{1cm} (4)

The minimal BRT is more applicable than the minimal BRS to safety, which requires avoiding danger not only at the final time but also at every intermediate point in time. In this paper, we compute the minimal BRS and then take its union over time to obtain the minimal BRT [26]:

$$\bar{A}(\mathcal{T}, t) = \bigcup_{s \in [t, 0]} A(\mathcal{T}, s), \hspace{1cm} (5)$$

if for all $s \in [t, 0]$, $A(\mathcal{T}, s) \neq \emptyset$.

B. Hamilton-Jacobi Reachability

Given $\mathcal{T}$, there exists a continuous function $l$, whose zero sublevel set represents the target set $[7]$, $\mathcal{T} = \{ z : l(z) \leq 0 \}$. Using this convention, this reachability problem can be formulated into an optimal control problem as below:

$$V(z, t) := \max_{u(\cdot)} l(\zeta(0; z, t, u(\cdot))), \hspace{1cm} (6)$$

which implies that the zero sublevel set of $V$ represents the minimal BRS: $A(\mathcal{T}, t) = \{ z : V(z, t) \leq 0 \}$.

The value function $V$ can be computed by solving an HJ PDE [7]. Suppose $f$ is Lipschitz continuous on $z$ and $l$ is also Lipschitz continuous, then $V$ is the unique viscosity solution [30] of the following HJ PDE:

$$\frac{\partial V}{\partial s} + \max_{u} \nabla V \cdot f(z, u) = 0, \hspace{0.5cm} V(z, 0) = l(z). \hspace{1cm} (7)$$

The optimal control of (6) is then given by

$$u^*(z, s) = \arg \max_{u} \nabla V(z, s) \cdot f(z, u). \hspace{1cm} (8)$$

The HJ PDE (7) is typically solved numerically using one of several Lax-Friedrichs numerical schemes, which involve an artificial dissipation term for approximating the $\max_{u} \nabla V \cdot f(z, u)$ term [31]-[33]. If the target set involves few or no sharp corners, one would expect less dissipation error in the numerical solution.

C. System Decomposition For Dimensionality Reduction

One common theme among decomposition methods such as [24], [25], [34] is representing the target set as an intersection of several other sets, computing BRSs from each of these sets, and then taking the intersection of these BRSs. However, when the target set is a union of sets, these methods produce inner-approximations of minimal BRSs and BRTs due to the leaking corner issue. Unfortunately, for safety applications, the minimal BRT represents the set of states that must be avoided, and therefore one would like to obtain its outer-approximation.

Lemma 1 Suppose $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. Define the two BRSs, $A(\mathcal{T}_1, t)$ and $A(\mathcal{T}_2, t)$, whose target sets are $\mathcal{T}_1$ and $\mathcal{T}_2$. Then, the union of $A(\mathcal{T}_1, t)$ and $A(\mathcal{T}_2, t)$ is an inner-approximation of the BRS:

$$A(\mathcal{T}, t) \supset A(\mathcal{T}_1, t) \cup A(\mathcal{T}_2, t) \hspace{1cm} (9)$$

Proof: If $z \in A(\mathcal{T}_1, t) \cup A(\mathcal{T}_2, t)$, then there exists $i \in \{1, 2\}$ such that for all $u(\cdot) \in U(\cdot)$, $\xi(0; z, t, u(\cdot)) \in \mathcal{T}_i$. This implies that $\xi(0; z, t, u(\cdot)) \in \mathcal{T}$ for all $u(\cdot)$, and so $z \in A(\mathcal{T}_1, t)$.

If $z \in A(\mathcal{T}, t)$, for all $u(\cdot) \in U(\cdot)$, $\xi(0; z, t, u(\cdot)) \in \mathcal{T}$. However, in general it is possible that there exists $u(\cdot)$ ($i = 1, 2$) such that $\xi(0; z, t, u(\cdot)) \in \mathcal{T}_1 \setminus \mathcal{T}_2$, and $\xi(0; z, t, u(\cdot)) \in \mathcal{T}_2 \setminus \mathcal{T}_1$. In this case, it is false that for $i = 1, 2$ that for all $u(\cdot) \in U(\cdot)$, $\xi(0; z, t, u(\cdot)) \in \mathcal{T}_i$. ■

In this paper, we define leaking corner errors to be the relative complement of $[A(\mathcal{T}_1, t) \cup A(\mathcal{T}_2, t)] \cap A(\mathcal{T}, t)$. The leaking corner error is a set of states that do not have obstacle avoidance controls for $\mathcal{T}$ although there exists each avoidance control for each of $\mathcal{T}_1$ and $\mathcal{T}_2$. Lemma 1 shows the leaking corner errors represent the non-conservative approximation errors.

D. Mixed-Implicit and Explicit Formulation

In this paper, we address the leaking corner problem while reducing system dimensionality for systems of a similar form as the mixed-implicit and explicit (MIE) formulation in [24], in which the state $z$ is written as $z = (x, y)$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Suppose the system dynamics may be written in the following form:

$$\dot{x} = b(y), \hspace{0.5cm} \dot{y} = h(y, u) \hspace{1cm} (10)$$

and the target set function may be written as:

$$l_1(x, y) = x - l_1^0(y) \hspace{0.5cm} \text{or} \hspace{0.5cm} l_2(x, y) = -x + l_2^0(y). \hspace{1cm} (11)$$

Note that $x = l_1^0(y)$ represents a boundary of the target set. Then, the solution of the HJ PDE in (7) is represented by reduced-order value functions $W_1$ and $W_2$ as follows:

$$V_1(x, y, t) = x - W_1(y, t), V_2(x, y, t) = -x + W_2(y, t), \hspace{1cm} (12)$$

where $W_1, W_2$ are solutions of

$$0 = \frac{\partial W_1}{\partial s} + \min_{u} \{ \nabla W_1 \cdot h(y, u) - b(y) \}, \hspace{1cm} (13a)$$

and

$$0 = -\frac{\partial W_2}{\partial s} + \max_{u} \{ \nabla W_2 \cdot h(y, u) - b(y) \}. \hspace{1cm} (13b)$$

and $W_1(y, 0) = l_1^0(y), W_2(y, 0) = l_2^0(y)$. The HJ PDE in (7) has now been converted to two reduced order PDEs in (13). Intuitively, $x = W_1(y, t)$ ($i = 1, 2$) represents the boundary of the BRS and (13) updates this boundary over time.

As in Lemma 1, if the target set is the union of the two sets represented by $l_1$ and $l_2$, the union of the two sets represented by $W_1$ and $W_2$ is an inner-approximation of the BRS by (9):

$$A(\mathcal{T}, t) \supset \{(x, y) \mid x \leq W_1(y, t) \text{ or } W_2(y, t) \leq x \}. \hspace{1cm} (14)$$

Note that in [24], (13a) and (13b) are solved independently of each other. Building on the MIE formulation, we will propose new coupled HJ PDEs that produce an outer-approximation of the minimal BRS to guarantee safety.
E. Problem Statement

Given a target set in the union form $\mathcal{T} = T_1 \cup T_2$, we aim to outer-approximate the minimal BRS with dimensionality reduction but still maintaining the safety guarantee. To achieve this, we will take the union of two sets, $\mathcal{A}_1, \mathcal{A}_2$, defined as follows:

$$\mathcal{A}_1(T_1, t) = \{ z \mid l_1(0; z, t, \tilde{u}_1(\cdot)) \in T_1 \},$$

$$\mathcal{A}_2(T_2, t) = \{ z \mid l_2(0; z, t, \tilde{u}_2(\cdot)) \in T_2 \},$$

where $\tilde{u}_1, \tilde{u}_2$ are chosen such that $\mathcal{A}(T, t) \subseteq \mathcal{A}_1(T_1, t) \cup \mathcal{A}_2(T_2, t)$. Note that union of an outer-approximation of the BRS over time is an outer-approximation of the BRT by (5). To compute $\mathcal{A}_1, \mathcal{A}_2$, we solve a set of coupled HJ PDEs sharing a common control, instead of solving two independent HJ PDEs.

In this paper, we first focus on the problem of leaking corners and ignore dimensionality reduction. We propose a design of $\tilde{u}_1$ and $\tilde{u}_2$ and coupled HJ PDEs for level set functions denoted $V_1^*$ and $V_2^*$ such that

$$\tilde{A}(T, t) = \tilde{A}_1(T_1, t) \cup \tilde{A}_2(T_2, t),$$

where $\tilde{A}_1(T_1, t) = \{ z \mid V_1^*(z, t) \leq 0 \}, i = 1, 2.$ (17)

Second, for dimensionality reduction, we consider the following system dynamics, slightly generalized from (10):

$$\dot{z} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} b(y, u) \\ h(y, u) \end{bmatrix},$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Given (18), we approximate $V_i^*$ in (17) and propose coupled HJ PDEs for level set functions $W_i^*$ and $W_2^*$ such that

$$\tilde{A}(T, t) \subseteq \tilde{A}_1(T_1, t) \cup \tilde{A}_2(T_2, t),$$

where $\tilde{A}_1(T_1, t) = \{ (x, y) \mid x \leq W_1^*(y, t) \}$, $\tilde{A}_2(T_2, t) = \{ (x, y) \mid W_2^*(y, t) \leq x \}.$ (19)

III. COUPLED HJ PDEs

We now propose (in Section III-A) coupled HJ PDEs whose solutions are level set functions $V_1^*$ and $V_2^*$ that satisfy (17), and (in Section III-B) coupled HJ PDEs whose solutions are $W_1^*$ and $W_2^*$ that satisfy (19).

A. Addressing Leaking Corners with Coupled HJ PDEs

We first define two level set functions whose values at the final time are given by $l_1$ and $l_2$, and using them, we reconstruct the exact level set function $V$ in (6). Denote

$$u^*(\cdot) := \arg \max_{u(\cdot)} \min \{ l(0; 0, z, t, u(\cdot)) \},$$

and note that point-wise optimal control $u^*$ is introduced in (8). Suppose $l_i$ ($i = 1, 2$) is a level set function that describes $T_i$, and

$$l(z) := \min \{ l_1(z), l_2(z) \}. $$

Consider two level set functions that share the same optimal control, $u^*$,

$$V_1^*(z, t) := l_1(0; 0, z, t, u^*(\cdot)),$$

$$V_2^*(z, t) := l_2(0; 0, z, t, u^*(\cdot)).$$

Lemma 2

$$V(z, t) = \min \{ V_1^*(z, t), V_2^*(z, t) \}$$

Proof:

$$V(z, t) = l(0; 0, z, t, u^*(\cdot)),$$

$$= \min \{ l_1(0; 0, z, t, u^*(\cdot)), l_2(0; 0, z, t, u^*(\cdot)) \},$$

$$= \min \{ V_1^*(z, t), V_2^*(z, t) \}$$

This implies that we can derive the exact value of $V$ using those two values, $V_1^*$ and $V_2^*$, which removes the leaking corner errors and achieves (17).

It remains for us to establish the corresponding HJ PDEs for $V_1^*$ and $V_2^*$. We first propose Theorem 1 that provides coupled HJ PDEs to compute $V_i^*$. Using this, we can get the exact BRS from the zero sublevel set of $V$ in (23).

Theorem 1 Suppose $u^*$ is given. $V_1^*$ and $V_2^*$ in (22) are viscosity solutions of

$$0 = \frac{\partial V}{\partial s}(z, 0) + \nabla V \cdot f(z, u^*) \quad \text{and} \quad V_1^*(z, 0) = l_1(z),$$

$$0 = \frac{\partial V}{\partial s}(z, 0) + \nabla V \cdot f(z, u^*) \quad \text{and} \quad V_2^*(z, 0) = l_2(z).$$

Proof: We refer to [30] for the definition of viscosity solutions. Without loss of generality, we focus on $V_1^*$ since the proof follows the same logic for $V_2^*$.

First we check the final value of the PDE:

$$V_1^*(z, 0) = l_1(0; 0, z, 0, u^*(\cdot)) = l_1(z).$$

Next, we show that if $\tilde{V} \in C^\infty$ and $V_1^* - \tilde{V}$ has a local maximum at a point $(z_0, t_0)$, that

$$\frac{\partial \tilde{V}}{\partial s}(z_0, t_0) + \nabla \tilde{V}(z_0, t_0) \cdot f(z_0, u^*(z_0, t_0)) \geq 0.$$

Suppose not, then the left term in (27) is strictly less than 0. Suppose $\xi(s)$ is a solution of

$$\dot{\xi}(s) = f(\xi(s), u^*(\xi(s), s)) \quad \text{and} \quad \xi(t_0) = z_0,$$

and assume that $f(\xi(s), u^*(\xi(s), s))$ is continuous on $s \in [t_0, 0]$. In the rest of the proof, we use $u^*(\xi(s), s)$ for compact notation. Then, there exists $h \in (0, -t_0)$ and $\theta > 0$ such that

$$\frac{\partial \tilde{V}}{\partial s}(\xi(s), s) + \nabla \tilde{V}(\xi(s), s) \cdot f(\xi(s), u^*(s)) \leq -\theta$$

for all $s \in [t_0, t_0 + h]$.

By (28), $V_1^*(\xi(t_0 + h), t_0 + h) = l_1(\xi(0)) = V_1^*(\xi(t_0), t_0)$.

Then, we arrive at a contradiction:

$$0 = V_1^*(\xi(t_0 + h), t_0 + h) - V_1^*(\xi(t_0), t_0) \leq -\delta h < 0.$$
difference schemes. By (8), \( u^* \) is derived from \( \nabla V \), and we use it to compute \( V_i^* \) at the next time discretization via numerical methods for solving (25). This numerical process is repeated over the desired time horizon.

**Algorithm 1** How to synthesize \( u^* \) and compute \( V_i^* \)

1. **Input:** Time discretization \( \Delta s \), number of time steps \( k \)
2. **Initialization:** \( V_1^* = l_1 \) and \( V_2^* = l_2 \)
3. For \( s \in \{0, -\Delta s, -2\Delta s, \ldots, -k\Delta s = t\} \) do
   4. Compute \( V(z, s) \) by (23) and \( \nabla V(z, s) \)
   5. Compute \( u^*(z, s) \) by (8)
   6. Update \( V_i^*(z, s) \) by (25)
   7. End for

**B. Coupled HJ PDEs with Dimensionality Reduction**

In this subsection, we propose coupled HJ PDEs in lower dimensions and make a conjecture that the proposed HJ PDEs provide a conservative approximation of the BRS and BRT. Suppose (18) holds. Our proposed formulation is based on Theorem 1 and is combined with the MIE formulation as introduced in Sec. II-D. Substituting \( x - W_1^*(y, t) \) and \( -x + W_2^*(y, t) \) for \( V_i^* \) and \( V_2^* \) in (25), we get

\[
0 = \frac{\partial V_i^*}{\partial s} + \nabla_y W_i^* \cdot h(y, u^*) - b(y, u^*), \quad (30)
\]

\[
0 = \frac{\partial V_2^*}{\partial s} + \nabla_y W_2^* \cdot h(y, u^*) - b(y, u^*), \quad (31)
\]

where \( u^* \) is as in (8). Notice that \( u^* \) is a function of \((x, y, s)\) not just a function of \((y, s)\).

Since \( u^* \) is a function of \((x, y, t)\), Equations (30) and (31) involve the state component \( x \). This implies that \( W_1^* \) and \( W_2^* \) must also have \( x \)-dependency. We must remove this dependency to obtain dimensionality reduction in the PDEs. In Conjecture 1, instead of \( u^* \), we present an approximate control \( u_1(y, t) \) and \( u_2(y, t) \) used in (32a) so that both dimensionality reduction and a safety guarantee might be achieved.

Define an approximate value function to \( V, V := \min\{x - W_1^*(y, t), -x + W_2^*(y, t)\} \), and denote \( \bar{u}(x, y, t) \) as the corresponding optimal control that maximizes the value \( V \) at \((x, y, t)\). We define \( u_1(y, t) := \bar{u}(W_1^*(y, t), y, t) \) and \( u_2(y, t) = u_2(W_2^*(y, t), y, t) \). The reason we pick controls at \((W_1^*(y, t), y, t)\) and \((W_2^*(y, t), y, t)\) is that the boundary of approximate BRS lies on \( x = W_1^*(y, t) \) or \( x = W_2^*(y, t) \), and that applying the optimal control on the boundary of the BRS is sufficient for avoiding the target set. This motivates us to propose the reduced-order HJ PDEs (32) below:

**Conjecture 1** Suppose (18) and (21) hold. Let \( W_1^* \) and \( W_2^* \) solve

\[
\begin{align*}
W_i^*(y, 0) &= l_i^0(y), \\
0 &= \frac{\partial W_i^*}{\partial s} + \nabla_y W_i^* \cdot h(y, u_i) - b(y, u_i), \quad i \in \{1, 2\}
\end{align*}
\]

(32a)

where

\[
\begin{align*}
u_i &= \bar{u}(W_i^*(y, s), y, s), \\
\bar{u}(x, y, s) &= \arg \max_u \nabla V(x, y, s) \cdot f(x, y, u), \\
\hat{V}(x, y, s) &= \min\{x - W_1^*(y, s), -x + W_2^*(y, s)\}.
\end{align*}
\]

(32b)

(32c)

(32d)

Then, (19) is satisfied.
with (23), and the little difference between the two zero-level sets is due to numerical dissipation errors discussed in [36]. Notice that the zero sublevel set of \( \min \{x - W_1, -x + W_2\} \) (pink) shows an inner-approximation of the BRS (red) and leaking corner errors, which follows (14).

As shown in Fig. 1, the zero sublevel set of \( \min \{x - W_1, -x + W_2\} \) (blue) shows an outer-approximation of the BRS that guarantees safety. This shows that the set derived by the JC MIE formulation is strictly an outer-approximation of the BRS.

We compare the computation times of the four HJ PDEs with respect to the number of grid points in each dimension. Among these data, the key point is that the JC MIE formulation with dimensionality reduction that solves (32) dramatically reduces computation time by comparing with the one for \( V \) solving (7) with respect to the grid size.

B. 5D Vehicle Obstacle Avoidance

In this subsection, we address a practical example (5D car model), whose minimal BRS and BRT have previously been considered to be intractable to compute directly, but we combine the JC MIE formulation and the decomposition method using self-contained systems (SCS) [26] to compute a conservatively approximate BRS and BRT in 3D that also provides an avoidance control trajectory.

The 5D car model has five states \((p^x, p^y, v, \theta, w)\) and moves in the 2D-plane. The dynamics are given by

\[
\begin{align*}
\dot{p}^x &= v \cos(\theta) & \dot{p}^y &= v \sin(\theta) \\
\dot{v} &= a & \dot{\theta} &= w & \dot{w} &= \alpha,
\end{align*}
\]

where \( p^x, p^y, v, \theta, \) and \( \omega \) represent the car’s \( x, y \)-position, speed, heading, and turn rate, respectively. The controls are \( a, \alpha \), which represent linear acceleration and angular acceleration, respectively. The control limits are \( \mathcal{U} = \{ \alpha | |\alpha| \leq 1, |\alpha| \leq 1 \} \). We define the target function as follows:

\[
l(p^x, p^y, v, \theta, \omega) := \max \{ l^B(p^x, p^y, v, \theta, \omega), l^P(p^x, p^y, v, \theta, \omega) \},
\]

where

\[
l^B(p^x, p^y, v, \theta, \omega) := \min \{ -p^x - l^B(v, \theta, \omega), -p^y - l^B(v, \theta, \omega) \},
\]

\[
l^P(p^x, p^y, v, \theta, \omega) := \begin{cases} 
4 & \text{if } -v < |\omega| < 0 \\
2(|v| - 4) - 2 & \text{if } 0 < |v| - 4 \leq \epsilon \\
-2 & \text{if } |v| - 4 < 0.
\end{cases}
\]

As \( \epsilon \to 0 \), the zero sublevel set of \( l \) converges to \( \{|x|, |y| \geq 2\} \cup \{|v| \geq 4\} \), which represents being the outside of a crossroad with excessive speed. Due to the continuity requirement of target set functions, we set \( \epsilon = 0.1 \).

The authors in [26] have shown that the BRS of a target set represented by the implicit surface function \( l \) (5D in this case) is the intersection of the BRSS of target sets represented by \( l^B \) and \( l^P \) respectively, each of which can be computed in 4D by SCS decomposition method. Since \( l^B \) and \( l^P \) are in a form of (11), using our JC MIE formulation, we obtain each 4D BRS by computing two 3D value functions by choosing \( x = p^x \) and \( x = p^y \) respectively, and \( y = (v, \theta, \omega) \) where \( x, y \) are defined in (18). In total, we compute four 3D value functions whose combination represents an approximate value function for the 5D BRS.

Fig. 3 compares the computation times for the BRS in 5D, 4D, and 3D state spaces when the number of discretizations of each state is \((51, 51, 51, 51, 61)\). The computation times in 3D (MIE+SCS), 4D (SCS), and 5D are 41.4 s, 40 min \((2402.1\ s)\), and 32.5 hours \((1.17 \times 10^5\ s)\), respectively. These computation times illustrate the exponential computational complexity, since increasing the dimensionality of the computation by one leads to a 53.2-fold increase in computation time, and 53.2 is approximately the number of grid points in each dimension of the computation grid.

Fig. 4 shows 3D slices of the approximate 5D BRT (red) and the target set (green). As in (5), we take the union of all approximate BRS over the time horizon and obtain the approximate BRT in 5D. Each column has a fixed \( v \) with varying \( \omega \), and each row has a fixed \( \omega \) with varying \( v \).

As the absolute value of \( v \) increases, the vehicle has more difficulty avoiding the obstacles, so the approximate BRT becomes bigger. On the other hand, if the initial \( v \) is zero, we get the biggest safe volume where the vehicle can avoid obstacles. Furthermore, observe the following symmetry which agrees with intuition: If \((p^x, p^y, v, \theta, \omega)\) is in the BRT, \((-p^x, -p^y, -v, \theta, \omega)\), \((-p^x, -p^y, v, \theta + \pi, \omega)\), and \((p^x, -p^y, v, -\theta, -\omega)\) are also in the BRT.

Fig. 5 shows two avoiding trajectories for different initial states, and the avoiding controls are given by (32b). In this figure, we add the target set (green) and the approximate BRT (red), whose \((v, \omega)\) is \((3.2, 2.83)\) that is the same \((v, \omega)\) of the two initial states. As shown in Fig. 5 (a), the initial state \((0.8, 0.8, 3.2, 0, 2.83)\) is outside of the approximate BRT, and the avoiding trajectory is safe along the time horizon. In Fig. 5 (b), the initial state \((1.2, 1.2, 3.2, 0, 2.83)\) is inside of the approximate BRT, and the avoiding trajectory is unsafe and collides with the obstacle at \(-0.55\ s\).

V. CONCLUSION AND FUTURE WORK

Reachability has various potential applications in robot safety and goal-reaching problems in manufacturing, automated vehicles, and air traffic management domains. In the area of system decomposition research for reducing computation complexity in reachability, this paper presents the joint-control MIE formulation, as a conjecture, that
approximates the HJ formulation in lower dimension by eliminating leaking corner issues. Our simulation results show that 1) solving coupled HJ PDEs in full dimension removes leaking corner errors, 2) solving coupled, lower-dimensional HJ PDEs provides a safety-guaranteed level set function with dramatically reduced computation time, thus allowing the computation of previously intractable higher dimensional BRSs and BRTs, 3) our MIE method can be applied with other dimensionality reduction methods, and 4) our method provides controllers that guarantee collision avoidance over a desired time horizon. With small modifications, our method can also be used to conservatively compute maximal BRSs of an intersection of target sets, a problem that also has leaking corner issues.

For future work, we will look for a mathematical proof to show why the coupled HJ PDEs provide the conservative approximation that guarantees safety.

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**REFERENCES**


